THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 5

§33) 1) b) Since $1-i=\sqrt{2}e^{-\frac{\pi}{4}i}$, we have

$$\mathrm{Log}(1-i)=\ln|1-i|+i\,\mathrm{Arg}(1-i)=\ln\sqrt{2}-\frac{\pi}{4}i=\frac{1}{2}\ln 2-\frac{\pi}{4}.$$

§33) 2) c) Since $-1+\sqrt{3}i=2e^{\frac{2\pi}{3}i}$, we have

$$\log(-1+\sqrt{3}i)=\ln|-1+\sqrt{3}i|=\ln 2+\left(\frac{2\pi}{3}+2n\pi\right)i=\ln 2+2\left(n+\frac{1}{3}\right)\pi i,$$

where $n\in\mathbb{Z}$.

§33) 3) Note that

$$\mathrm{Log}(i^3)=\mathrm{Log}(-i)=\mathrm{Log}(e^{-\frac{\pi}{2}i})=-\frac{\pi}{2}i;$$
$$3\,\mathrm{Log}(i)=3\,\mathrm{Log}(e^{\frac{\pi}{2}i})=3\left(\frac{\pi}{2}i\right)=\frac{3\pi}{2}i\neq-\frac{\pi}{2}i=\mathrm{Log}(i^3).$$

Hence $\mathrm{Log}(i^3)\neq 3\,\mathrm{Log}(i)$.

§33) 7) Since rectangular coordinates and polar coordinates are related by

$$r=\sqrt{x^2+y^2}\text{ and }\theta=\tan^{-1}\left(\frac{y}{x}\right),$$

we have

$$\log(z)=\ln\sqrt{x^2+y^2}+i\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\ln(x^2+y^2)+i\tan^{-1}\left(\frac{y}{x}\right).$$

Let $u(x,y)=\frac{1}{2}\ln(x^2+y^2)$ and $v(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$. Note that

$$u_x=\frac{x}{x^2+y^2}=v_y\text{ and }u_y=\frac{y}{x^2+y^2}=-v_x.$$

Moreover, since u_x,u_y,v_x,v_y are continuous and satisfy the CR equation on its domain, the function $\log(z)$ is analytic with

$$\log(z)=u_x+iv_x=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}=\frac{x-iy}{x^2+y^2}=\frac{\overline{z}}{|z|^2}=\frac{1}{z}.$$

Remark: Note that the function $\tan^{-1}\left(\frac{y}{x}\right)$ is well-defined only when $x\neq 0$. Furthermore, since $\tan^{-1}\left(\frac{y}{x}\right)$ is a multi-valued function, we have to specify the choose the values in such a way that it agrees with the branch of the logarithm function. To know more about it, you may see `https://en.wikipedia.org/wiki/Complex_logarithm` and `https://en.wikipedia.org/wiki/Atan2`.

§33) 12) For $z\neq 1$,

$$u(x,y)=\mathrm{Re}[\log(z-1)]=\mathrm{Re}[\log((x-1)+yi)]=\ln|(x-1)+yi|=\frac{1}{2}\ln[(x-1)^2+y^2].$$

Note that for any $z\neq 1$, the function $u(x,y)$ is the real part of the analytic function $\log_\alpha(z-1)$, where the branch α is choosen so that z lies inside the domain of $\log_\alpha(z-1)$. Therefore, $u(x,y)$ satisfies the Laplace equation.

§34) 1) Let $z_k = r_k e^{i\theta_k}$, where $k = 1, 2$ and $\theta_k \in (-\pi, \pi]$. In particular, we have $\theta_1 + \theta_2 \in (-2\pi, 2\pi]$. We have three possibilities:

- if $\theta_1 + \theta_2 \in (-2\pi, -\pi]$, then we have

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(r_1 r_2 e^{i\theta_1 + \theta_2}) = \ln r_1 r_2 + i(\theta_1 + \theta_2 + 2\pi) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2\pi i.$$

- if $\theta_1 + \theta_2 \in (-\pi, \pi]$, then we have

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(r_1 r_2 e^{i\theta_1 + \theta_2}) = \ln r_1 r_2 + i(\theta_1 + \theta_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2).$$

- if $\theta_1 + \theta_2 \in (\pi, 2\pi]$, then we have

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(r_1 r_2 e^{i\theta_1 + \theta_2}) = \ln r_1 r_2 + i(\theta_1 + \theta_2 - 2\pi) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2) - 2\pi i.$$

To conclude, we have

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}(r_1 r_2 e^{i\theta_1 + \theta_2}) = \ln r_1 r_2 + i(\theta_1 + \theta_2 + \pi) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2N\pi i,$$

where N has one of the values $0, \pm 1$.

§36) 1) b)

$$\begin{aligned}\frac{1}{i^{2i}} &= i^{-2i} \\ &= \exp\left[-2i \log(i)\right] \\ &= \exp\left[-2i(\ln|i| + i\arg(i))\right] \\ &= \exp\left[-2i\left(\left(\frac{\pi}{2} + 2n\pi\right) i\right)\right] \\ &= \exp[(4n+1)\pi],\end{aligned}$$

where $n \in \mathbb{Z}$.

§36) 2) a) The principal value of $(-i)^i$ is given by

$$(-i)^i = \exp(i\,\mathrm{Log}(-i)) = \exp\left[i\left(\ln|-i| - \frac{\pi}{2} i\right)\right] = \exp(\frac{\pi}{2})$$

§36) 6) For any $z \neq 0$ and $a \in \mathbb{R}$,

$$|z^a| = |\exp(a \log z)| = |\exp(a \ln|z| + ia\arg(z))| = \exp(a\ln|z|) = |z|^a$$

§36) 9) If $f'(z)$ exists, we have

$$\frac{d}{dz} c^{f(z)} = \frac{d}{df(z)} c^{f(z)} \frac{d}{dz} f(z) = c^{f(z)} f'(z) \log c.$$